TIDAL DYNAMICS IN COASTAL AQUIFERS

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for the award of the degree of

Master of Philosophy

by

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Declaration

This work has not previously been submitted for a degree or diploma in any university. To the best of my knowledge and belief, the Thesis contains no material previously published or written by another person except where due to reference is made in the Thesis itself.

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Liza Hhih-Ting Teo
February 2003
List of Publications

A. Journal papers:


B. Conference papers:


Abstract

The prediction of coastal groundwater movement is necessary in coastal management. However, the study in this field is still a great challenge due to the involvement of tidal-groundwater interactions and the phenomena of hydrodynamic dispersion between salt-fresh water in the coastal region. To date, numerous theories for groundwater dynamic have been made available in analytical, numerical and also experimental forms. Nevertheless, most of them are based on the zeroth-order shallow flow, i.e. Boussinesq approximation.

Two main components for coastal unconfined aquifer have been completed in this Thesis: the vertical beach model and the sloping beach model. Both solutions are solved in closed-form up to higher order with shallow water parameter ($\epsilon$) and tidal amplitude parameter ($\alpha$). The vertical beach solution contributes to the higher-order tidal fluctuations while the sloping beach model overcomes the shortcomings in the existing solutions.

From this study, higher-order components are found to be significant especially for larger value of $\alpha$ and $\epsilon$. Other parameters such as hydraulic conductivity ($K$) and the thickness of aquifer ($D$) also affect the water table fluctuations. The new sloping solution demonstrated the significant influence of beach slope ($\beta$) on the water table fluctuations. A comprehensive comparison between previous solution and the present sloping solution have been performed mathematically and numerically and the present solution has been demonstrated to provide a better prediction.
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List of Symbols

\( A \)  Tidal amplitude
\( D \)  Average height of the water table
\( H \)  Non-dimensional water table level
\( h \)  Tide-induced water table level
\( K \)  Hydraulic conductivity
\( L \)  Linear decay length in the \( x \) direction
\( n_e \)  Effective porosity
\( p \)  Water pressure
\( T \)  Period of tide
\( x \)  Horizontal axis measured positive inland from origin
\( z \)  Elevation which measured from the rigid porous medium
\( \phi \)  Potential head \( (\phi = z + p/\rho g) \)
\( \alpha \)  Ratio of the maximum tidal variation \( (\alpha = A/D) \)
\( \beta \)  Slope of the beach \( (0^\circ > \beta > 90^\circ) \)
\( \varepsilon \)  Shallow water parameter \( (\varepsilon = \frac{Dn_e\omega}{2K}) \)
\( \varepsilon_N \)  The perturbation parameter in Nielsen (1990) \( (\varepsilon_N = \frac{A}{L} \cot(\beta)) \)
\( \rho \)  Density of fluid
\( \omega \)  Tidal wave frequency \( (2\pi/T) \)
Chapter 1

Introduction

1.1 General Background

Over fifty percent of the earth populations depend daily on groundwater for drinking, and it is one of our most important sources of irrigation. Groundwater can be found underground in cracks and spaces in soil, sand and rocks. The area where water fully fills spaces underground is called the saturated aquifer zone. The top of this aquifer zone is called the water table. The water table may be deep or shallow; and may rise or fall depending on many factors. The layers of soils and rocks in aquifer are permeable due to the large connected spaces, that allow water to flow through. The speed at which groundwater flows depends on the size of the spaces in the soil or rock and how well the spaces are connected.

Groundwater can be polluted easily by human activities or by nature. The material above the aquifer is often permeable, allowing pollutants to penetrate into the ground, polluting the groundwater. Runoff from landfills, septic fields, leaky underground gas tanks, fertilizers and pesticides are among the typical factors that can pollute groundwater. In some areas such as coastal zones, groundwater can be polluted by the intrusion of salt water from the ocean induced by tidal actions (Dagan and Zeitoun, 1998).

In general, groundwater is categorised into inland groundwater and coastal groundwater. The latter is more complicated because it involves the phenomena of hydrodynamic dispersion, which is the combination of mechanical dispersion and molecular diffusion. This phenomenon of hydrodynamic dispersion will be discussed in detail later.

1.1.1 Definitions of aquifers

The word “aquifer” derives from the Latin words, “aqua” and “ferre”. “Aqua” has the meaning “water” and “ferre” has the meaning “bear”. Therefore, “aquifer” defines the geological material underground that has the ability to store or transport groundwater.
According to Bear (1972), aquifer stands for a porous geological medium formation that contains water and permits water to move through it. Whether a geological formation may be referred to as an aquifer or not, depends on its ability to store and transport water relative to other formations in the vicinity.

For a geological formation contains water, but it is incapable of transmitting water in appreciable quantities, such as in a clay and mud is not aquifer, instead it is known as aquiclude—a combination of words “aqua” and “exclude” (Bear, 1972).

1.1.2 Classifications

The geological structure underlying aquifer systems consists of several types of aquifer. Figure 1.1 shows a typical geological aquifer formation.

![Geological aquifer formations](Bear, 1972)

In general, aquifers are classified in five different categories (Bear, 1972);

- **Confined aquifer**—A confined aquifer is bounded by impermeable layers on the top and bottom aquifer. Across these layers, there is negligible flow. The saturated part is the full thickness of the aquifer, and the piezometric head in it is higher than the lower surface of the confining layer.

- **Artesian aquifer**—A portion of a confined aquifer in which the piezometric surface is not only above the ceiling of the aquifer, but also above ground. In an artesian aquifer, the water will directly flow out from wells without pumping.
• **Phreatic aquifer**—An aquifer that is bounded from above by a phreatic surface is called a phreatic, or unconfined aquifer, or a water table aquifer. Lowering the piezometric surface in a confined aquifer by over-pumping may result to a phreatic aquifer.

• **Perched aquifer**—A special case of a phreatic aquifer is the perched aquifer. This is a phreatic aquifer of limited area extent, formed on a semi-pervious, or impervious, layer that is present between a persistent water table of a phreatic aquifer and ground surface.

• **Leaky aquifer**—A leaky phreatic aquifer is a phreatic aquifer that is bounded from below by a semi-pervious layer and behaves as a "semi-pervious membrane" through which leakage out of or into the phreatic aquifer from an underlying saturated region is possible.

Figure 1.1 illustrates a vertical cross-section with different type of aquifers. Portions of aquifers A, B, and C are leaky, with directions and rates of leakage determined by the elevations of the piezometric surface of each of these aquifers. The boundaries between the various confined and unconfined portions may vary with time, as a result of fluctuations in the piezometric surfaces and the water table (Bear, 1972).

### 1.2 Significance of The Study

Coastal aquifers constitute an important source for water, as do inland aquifers. In Australia, about 75% of the population lives within a few kilometers off the coast. The coastal zone is used for activities such as settlement, industry and agriculture. Due to the increase in development in coastal regions, it is important to ensure proper coastal management and coastal protection techniques. Without proper care and management, problems such as beach erosion, saltwater intrusion and contamination of coastal aquifer may arise.

Tidal dynamics is a fundamental part of coastal hydrology. In general, tides are movements of the oceans set up by gravitational effects of the sun and the moon in relation to the earth. They move in harmony with the gravitational forces of the sun and the moon, with an additional gyratory motion imparted by the earth’s rotation (Russell and MacMillan, 1952). As a tidal wave propagates through an aquifer, friction causes a loss of energy, which is manifested as a decreasing head condition (Sun, 1997). The tidal regime around Australia, as indicated by the pattern of co-tidal lines can be seen from Figure 1.2.

Tides lead to regular changes in the level of the sea along the coast with respect to time, and affect the change of groundwater profile on the coast. The phenomenon of tidal effects on groundwater motions has been investigated intensively by many researchers (Philip, 1973; Knight, 1982). Among these, Knight (1982) found that groundwater level is related to the root-mean-square of water level and the fluctuation period.
As the tide fluctuates from the coastline boundary, the sea water intrusion may be significant, especially when the mean tide is higher than the groundwater level on shore. The fluctuating amplitude of the groundwater table decays with distance away from the coastline and there is a phase shift in the fluctuation with respect to time. Coastal groundwater has a more complicated behavior compared to inland groundwater due to the aforementioned tidal actions, ocean wave actions and other factors which significantly influence the fluctuation of groundwater.

### 1.3 Outlines of Thesis

The objective of this study is to develop a series of mathematical models to investigate the tide-induced water table fluctuations in coastal aquifers. Two analytical solutions will be derived for vertical and sloping beaches. Both models consist of unconfined aquifers, where the material above the aquifer is porous (permeable), but is impermeable on the base. The models also include tidal oscillations that have a significant impact on the overall fluctuations of groundwater, especially close to the beach front.

Chapter 2 contains a detailed literature review. The literature review includes relevant research work for the relevant theoretical models. The contributions and shortcoming of previous analytical studies will be summarised in this chapter.
In Chapter 3, a new analytical, closed-form solution for the tidal fluctuations in adjacent to a vertical beach will be derived. The new solution provides a higher-order expansion of the problem. With the new solution, the effects of higher-order components and important physical parameters on the tidal fluctuations in a coastal aquifer will be investigated.

In Chapter 4, a new analytical solution for a sloping beach will be derived. The new solution overcomes the shortcoming of the previous study and provides a better understanding of the phenomenon of the tidal dynamics. Based on the new solution, the influences of the slope on the tidal fluctuation will be the main concern.

Finally, the conclusions of this study are presented in Chapter 5 with suggestions for future research.
Chapter 2

Literature Review

Groundwater dynamics within a sandy beach is important for erosion control, saltwater intrusion and biological activities. Ocean tide-induced water table fluctuations decay inland from the shoreline, and have an average level different to the mean sea level. The primary concern in many groundwater investigations is in the finding the variation in water table height with respect to different surrounding factors.

Groundwater models are commonly investigated using either analytical or numerical solutions. Analytical methods are superior, but numerical methods are often applicable where problems are too complicated to be described accurately with analytical solutions. Analytical solutions have the advantage that they offer a general understanding of solution behaviours.

The major difficulty in solving equation describing tidal fluctuations in coastal aquifers analytically has been the mathematical representation of the non-linear kinematic boundary conditions. The most commonly used method for a non-linear boundary value problem is a perturbation series. In this chapter, the previous analytical approaches are reviewed under two headings “Vertical beaches” and “Sloping beaches”.

2.1 Vertical Beaches

To simplify the problem, the case of a vertical beach has been considered as the first analytical approximation. The coastal groundwater is assumed to begin from the vertical boundary and continue inland.

Dagan (1967) investigated free surface aquifer flow up to second-order by using matching techniques. He demonstrated that the Boussinesq equation, resulting from a small parameter expansion, will yield the first-order free surface equation. Through using a matching technique to get the second-order solution, Dagan (1967) found that the inner expansion is valid beyond a certain distance from the outflow face and the flow in the zone adjacent to the outflow face is described by an outer expansion which matched with the inner expansion.
Following a similar procedure to Dagan (1967), Parlange et al. (1984) considered the flow in a porous medium. Parlange et al. (1984) pointed out that the second-order linearised solution could describe the groundwater free surface elevation, and is believed to be adequate when the amplitude of the motion is comparable to the time average depth of the groundwater. They concluded that the inclusion of second order free surface flow correction to the Boussinesq equation (i.e., $\alpha^2$, where $\alpha = A/D$, $A$ is the amplitude of tidal fluctuation, $D$ is the thickness of aquifer) is more accurate to predict the water level, compared with the first-order solution.

Jiao and Tang (1999) proposed a one-dimensional analytical solution of tidal fluctuations in a confined aquifer including leakage effects. Their solution is based on the three assumptions introduced by Hantush and Jacob (1955). The assumptions included that the head in the layer supplying the leakage is constant; the permeability contrast in the semi-pervious layer and aquifers is large enough to provide vertical flow in the semi-confining bed and horizontal flow in the aquifers respectively. In addition, the solution also ignored the storage of the semi-confining unit. However, as pointed out by Volker and Zhang (2001), Li et al. (2001) and Jeng et al. (2002), their solutions may be unrealistic and inappropriate for the case when the leakage is greater than unity.


Li and Jiao (2001) believed that the leakage is generally more important than the storage. They derived another analytical solution for a confined leaky aquifer that indicated that both the storage and leakage for the semi-permeable layer play important roles in the groundwater head fluctuations in the confined aquifer. They stated that the impact of storage on groundwater head fluctuations changes with leakage.

Assuming a sharp interface between fresh and salt water within a coastal aquifer in a circular island, Wang and Tsay (2001) obtained an analytical solution, which determines the piezometric head movement of steady and unsteady components in terms of large and small time scales. The tide-induced fluctuation amplitude generally decays in distance with a parameter consisting of hydraulic conductivity, storage coefficient, and thickness of aquifer and tidal period. This solution is applicable to confined and unconfined aquifers, with only freshwater flow or interfacial flow, and does not consider the capillarity effect.

In the formulation of unconfined groundwater flow problems, it is often assumed that the upper boundary of the flow domain is a free surface, which is a sharp boundary between saturated and dry aquifer material. This assumption is only ideal, convenient and often not suitable for practical purpose. In fact, the upper boundary of groundwater table is a diffuse transition zone, consisting of partly saturated material instead of a sharp air-water interface (Parlange and Brutsaert, 1987). This partially saturated zone above the groundwater table is known as the capillary fringe.
The conventional Boussinesq equation does not include the capillary effect. At the coastal boundary \((x=0)\) in a vertical boundary analysis, the capillarity effect is negligible (Barry et al., 1996). However, the capillarity correction to the groundwater movement is important for under high frequency flows (Parlange and Brutsaert, 1987, Barry et al., 1996, Li et al., 1997a).

Parlange and Brutsaert (1987) demonstrated that the Boussinesq equation of hydraulic groundwater theory can be corrected by the inclusion of capillary effects above the groundwater table. They concluded that the capillarity correction is important for times scale are smaller than the capillary time scale, \(B/K\), where \(B\) is the capillary parameter and \(K\) is the capillarity conductivity.

Using Parlange and Brutsaert’s (1987) capillary correction, Barry et al. (1996) derived a perturbation solution that modeled the behaviour of the phreatic surface up to second order by adopting a perturbation technique applied to the free surface condition. However, the analytical solution proposed by Barry et al. (1996) is only solved up to second order in \(\alpha\), which may be insufficient for the case of large tidal amplitudes. Using the second-order solution, Barry et al. (1996) further derived the time averaged mean square of the true phreatic surface height and compared it with the solution without capillary effect and a numerical solution. Capillarity effects provide a mechanism for the propagation in land of high-frequency sea level oscillations (Barry et al., 1996). Recently, Teo et al. (2002) further extended Barry et al. (1996) to order \(\alpha^3\), and found insignificant differences between the two solutions of \(O(\alpha^2)\) and \(O(\alpha^3)\).

All previous investigations have been limited to the zero-order shallow water expansion (\(O(1)\)), i.e., the Boussinesq equation. Such approximations are limited to the case, \(\varepsilon = \frac{L}{D} << 1\), where the thickness of the aquifer \((D)\) is much less than the decay length \((L)\). In this study, we will further derived an analytical solution up to order \(\varepsilon^2\), and investigate the influence of the higher-order components. This will be detailed in Chapter 3.

2.2 Sloping Beaches

Since the boundary condition for a sloping beach is more complicated than the case of a vertical beach, only a few researchers have attempted to solve the problem analytically (Nielsen, 1990; Jeng et al., 2003).

Nielsen (1990) may be the first to derive an analytical solution to describe the aquifer behaviour adjacent to a slopping beach. The perturbation parameter Nielsen (1990) is \(\varepsilon_N = \frac{A}{L} \cot \beta\) [it is \(\varepsilon\) in Nielsen (1990)], where \(\beta\) is the slope of the beach, and \(L\) is declaying length. The beach forms an angel \(\beta\) with the horizontal. Based on the analytical solution, Nielsen (1990) concluded that the inland over height due to the beach slope is of the order of magnitude of \(\frac{A}{2L} \cot \beta\). Nielsen (1990) also concluded that the asymptotic inland over height due to the non-linearity in the Boussinesq equation is of the order \(A^2/4D^2\).

With the definition of the perturbation parameter used in Nielsen (1990), it is obvious that the perturbation parameter restricts the applicable range of his model.
Also, Nielsen’s solution is only a linear solution, when it is reduced to the special case of a vertical beach by inserting $\beta = 90^\circ$ into his solution. Furthermore, Nielsen (1990) solution missed out several terms in the higher-order components. The shortcoming of Nielsen’s (1990) approximation was overcome by Jeng et al. (2003) by introducing a new perturbation parameter. These analytical solutions will be elaborated in Chapter 4.

2.3 Summary

This chapter has presented a literature review of tidal dynamics in coastal aquifers studies. The review covered the previous analytical studies done for both vertical and slopping beaches. Since the aim of this study is to develop a more advanced mathematical models for such a problem, the gaps between current knowledge and the proposed study are summarised here:

(1) Previous analytical solutions for both vertical and sloping beaches have been based on the Boussinesq equation, which is the zero-order of the shallow water expansion. A higher-order approximation is desired to provide a better understanding.

(2) The perturbation parameter used in the previous study for a sloping beach has been limited to a few particular slopes. An appropriate perturbation approximation is required for general cases.

To have a better understanding of the tidal dynamics in coastal aquifers, we will derive higher-order approximations to fill in the gaps mentioned above. The details of the new analytical solution will be presented in the following chapters.
Chapter 3

Tidal Dynamics in a Vertical Beach

Groundwater movements near the coast are of great interest in the management of coastal aquifers. In particular, an accurate prediction of dynamic groundwater hydraulics in coastal zones is required to improve coastal management. To simplify the problem, the case of a vertical beach has been used as the first approximation, especially in mathematical modelling (Dagan, 1967, Parlange et al., 1984; Barry et al., 1999). All previous investigations have been limited to the zeroth-order shallow water expansion, i.e., the Boussinesq equation.

In this chapter, a new analytical solution for tidal dynamics adjacent to a vertical beach is developed. First, the boundary value problem is set-up. Then, the general solution for tidal fluctuation is formulated in Section 2.2. Based on this solution, the effects of higher-order components and other physical parameters (such as the hydraulic conductivity and mean thickness of the aquifer) on the water table fluctuations are investigated.

3.1 Boundary Value Problem

3.1.1 Basic assumptions

The basic assumptions are made, which the fluid is

- incompressible,
- irrotational,
- inviscosity, and
- a sharp boundary exist in between the two miscible fluids in the vertical direction.
The mass exchange in between the sea and fresh water is small due to shallow water interactions and small tidal frequency. This ideal assumption eliminates the inclusion of density effect in between two miscible fluids. When the fluid is incompressible irrotational and inviscosity, the potential head in the fluid satisfies the Laplace equation.

3.1.2 Problem set-up

The phenomenon of ocean tides incident at a vertical beach is considered, as depicted in Figure 3.1. The tides will cause fluctuations of the water table in a coastal aquifer. The origin of the horizontal $x$-axis is the intersection of the mean sea level and the beach face, and $x$ extends positively inland. The $z$-axis is measured positive upward from the rigid porous medium.

Since the assumptions are made that fluid is incompressible, irrotational and the existing head in the fluid satisfies conservation of mass. This leads to Laplace equation (Bear and Verruijt, 1987):

$$\phi_{xx} + \phi_{zz} = 0,$$

where the potential head ($\phi$) is defined by $\phi = z + p/\rho g$, in which $z$ is the elevation, $p$ is the water pressure and $\rho$ is the density of fluid.

Equation (3.1) is solved subject to the following boundary conditions.

(a) Bottom Boundary Condition ($z = 0$):

At the bottom of the aquifer ($z = 0$), no vertical flows occurs at the horizontal rigid impermeable bottom.

$$\phi_z = 0, \quad \text{at} \quad z = 0.$$  \hspace{1cm} (3.2)

(b) Free Surface Boundary Condition ($z = h$):

At the free surface of the water table, the potential head ($\phi$) should be equal to the tide-induced water table fluctuation ($h$),

$$\phi = h, \quad \text{at} \quad z = h.$$  \hspace{1cm} (3.3)
In addition we have the kinematic boundary condition along the free surface as 
(de Marsily, 1986),
\[ n_e \frac{\partial \phi}{\partial t} = K \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] - K \frac{\partial \phi}{\partial z}, \quad \text{at} \quad z = h, \quad (3.4) \]
where \( n_e \) is the effective porosity of the soil layer and \( K \) is the hydraulic conductivity.

(c) Initial Boundary Condition \((x = 0)\):
At the interface of the ocean and land \((x = 0)\), the water table level is equal to the ocean wave oscillation, i.e.,
\[ \phi = h(0, t) = D(1 + \alpha \cos \omega t), \quad \text{at} \quad x = 0, \quad (3.5) \]
where \( \alpha = A/D \) is a dimensionless amplitude parameter, representing the ratio of the maximum tidal variation, \( A \), to the average height of the water table, \( D \). \( \omega \) is the wave frequency defined as \( 2\pi/T \), where \( T \) is the period of tide.

(d) Lateral Boundary Condition \((x \rightarrow \infty)\):
Since the influence of ocean tide on the water table fluctuation is only limited to the region near the coastline, the tide-induced water table fluctuation will vanish far away from the beach. That is, the gradient of water table fluctuation tend to zero, i.e.,
\[ \phi_x = 0, \quad \text{as} \quad x \rightarrow \infty. \quad (3.6) \]

3.1.3 Non-dimensional equations
The boundary value problem can be solved analytically in the shallow water limit. To simplify the problem and mathematical procedure, the following non-dimensional parameters are introduced:
\[ X = \frac{x}{L}, \quad Z = \frac{z}{D}, \quad H = \frac{h}{D}, \quad \Phi = \frac{\phi}{D}, \quad T = \omega t, \quad L = \sqrt{\frac{2KD}{n_e \omega}}, \quad (3.7) \]
where \( L \) is known as the linear decay length and is the significant length scale in the \( x \) direction. Here we consider shallow water flows and hence define the shallow water parameter, \( \varepsilon \), as
\[ \varepsilon = \frac{D}{L} = \sqrt{\frac{D n_e \omega}{2K}}. \quad (3.8) \]
The shallow water parameter is less than unity for most coastal aquifers. It is noted that the magnitude of \( \varepsilon \) depends on the values of \( \omega, K, n_e \) and \( D \). This implies that the higher-order solution will be more important with higher frequency (large \( \omega \)), low hydraulic conductivity (\( K \)) and large \( D \). This phenomenon will be demonstrated in the numerical examples in Section 3.4.

Using the shallow water parameter as the first perturbation parameter, the potential head \( (\Phi) \) and water table level \( (H) \) are expanded in powers of \( \varepsilon \),
\[ \Phi = \sum_{n=0}^{\infty} \varepsilon^n \Phi_n, \quad (3.9a) \]
\[ H = \sum_{n=0}^{\infty} \varepsilon^n H_n. \quad (3.9b) \]

Substituting (3.9a) and (3.9b) into the boundary value problem (3.1)–(3.6), results in the following boundary value problems
\[ O(1) : 2H_0T = (H_0H_{0X})_X, \quad (3.10a) \]
\[ O(\varepsilon) : 2H_1T = (H_0H_1)_{XX}, \quad (3.10b) \]
\[ O(\varepsilon^2) : 2H_2T = (H_0H_2)_X + \frac{1}{3} (H_0^3H_{0XX})_{XX}, \quad (3.10c) \]
\[ H_0(0, t) = 1 + \alpha \cos(T), \quad H_1(0, t) = H_2(0, t) = \cdots = 0, \quad (3.10d) \]
\[ H_0X(\infty, T) = H_1X(\infty, T) = H_2X(\infty, T) = 0. \quad (3.10e) \]

The mathematical details of the above boundary value problems are given in Appendix.

### 3.2 General Solution

#### 3.2.1 Zeroth-order approximation

From (3.10a), the governing equation for the zeroth-order approximation is given as:
\[ 2H_0T = (H_0H_{0X})_X. \quad (3.11) \]

This is the Boussinesq equation and is non-linear.

In order to solve this non-linear equation, the second perturbation parameter, \( \alpha = A/D \), is used to linearise (3.11). In general, the amplitude of tide waves \( (A) \) is small compared with the thickness of aquifer \( (D) \). Thus, \( \alpha \) is an appropriate perturbation parameter.

Then, zeroth-order tidal fluctuations can be expressed as
\[ H_o = 1 + \sum_{n=1}^{\infty} \alpha^n H_{0n}. \quad (3.12) \]

Using (3.12), equation (3.11) can be expanded and sorted in different orders of amplitude parameter, \( O(\alpha^i) \):
\[ O(\alpha) : \quad 2H_{01T} = H_{01XX}, \quad (3.13a) \]
\[ O(\alpha^2) : \quad 2H_{02T} = H_{02XX} + \frac{1}{2} (H_{01})_{XX}, \quad (3.13b) \]
\[ O(\alpha^3) : \quad 2H_{03T} = H_{03XX} + (H_{02}H_{01})_{XX}, \quad (3.13c) \]

with boundary conditions
\[ H_{01}(0, T) = \cos(T), \quad H_{02}(0, T) = H_{03}(0, T) = 0, \quad (3.14a) \]
\[ H_{01X}(\infty, T) = H_{02X}(\infty, T) = H_{03X}(\infty, T) = 0. \quad (3.14b) \]
Solution of $O(\alpha)$

The boundary value problem of $O(\alpha)$ is

\begin{align}
2H_{01T} & = H_{01XX}, \\
H_{01}(0,T) & = \cos(T), \\
H_{01X}(\infty,T) & = 0.
\end{align}

(3.15a) (3.15b) (3.15c)

With the linear solution of tidal fluctuation, $H_{01}$, can be written as

$H_{01}(X,T) = \exp(-X)\cos(T-X)$.

(3.16)

Solution of $O(\alpha^2)$

The boundary value problem of $O(\alpha^2)$ is summarised as

\begin{align}
2H_{02T} & = H_{02XX} + \frac{1}{2}(H_{01}^2)_{XX}, \\
H_{02}(0,T) & = 0, \\
H_{02X}(\infty,T) & = 0.
\end{align}

(3.17a) (3.17b) (3.17c)

Substituting (3.16) into (3.17a), we have

$2H_{02T} = H_{02XX} + \frac{1}{2}\exp(-2X) + \frac{1}{2}\exp(-2X)\cos 2(T - X)$.

(3.18)

The homogeneous solution of (3.18) is

$H_{02h} = C \exp\left(-\sqrt{2}X\right)\cos\left(2T - \sqrt{2}X\right)$,

(3.19)

where $C$ is an unknown coefficient to be determined by the initial boundary condition at $X = 0$.

The particular solution of time-independent term in (3.18) is

$H_{02P_1} = \frac{1}{4}(1 - \exp(-2X))$,

(3.20)

and the particular solution of the time-dependent term in (3.18) is

$H_{02P_2} = -\frac{1}{2}\exp(-2X)\cos 2(T - X)$.

(3.21)

Then $C = \frac{1}{2}$ by applying the initial boundary condition to the completed solution (i.e., sum of homogeneous and particular solutions). Thus, the final solution is

$H_{02} = \frac{1}{4}(1 - \exp(-2X)) + \frac{1}{2}\exp(-\sqrt{2}X)\cos(2T - \sqrt{2}X)
- \frac{1}{2}\exp(-2X)\cos 2(T - X)$.

(3.22)

Note that the above solution of $O(\alpha)$ and $\alpha^2$ are identical to Paralange et al. (1984).
**Solution of O(α³)**

The boundary value problem of $O(\alpha³)$ is summarised as

$$
2H_{03T} = H_{03XX} + (H_{01}H_{02})_{XX},
$$

$$
H_{03}(0, T) = 0,
$$

$$
H_{03X}(\infty, T) = 0.
$$

Following a similar procedure to $O(\alpha²)$, we have the solution at $O(\alpha³)$ as

$$
H_{03} = \frac{X}{8} \exp(-X) [\cos(T - X) - \sin(T - X)]
$$

$$-\frac{1}{32} \exp(-X)[3\cos(T - X) + 4\sin(T - X)]
$$

$$+\frac{1}{32} \exp(-3X)[3\cos(T - 3X) + 4\sin(T - 3X)]
$$

$$+\frac{1}{8} (4 + 3\sqrt{2}) \exp\left(-\sqrt{2} + 1\right) \cos(3T - (\sqrt{2} + 1)X)
$$

$$-\frac{1}{8} (4 + 3\sqrt{2}) \exp\left(-\sqrt{3}X\right) \cos(3T - \sqrt{3}X)
$$

$$+\frac{3}{4} \exp(-3X) \cos(3(T - X) - \exp\left(-\sqrt{3}X\right) \cos(3T - \sqrt{3}X)].
$$

A term proportional to $X$ appears in the solution of $O(\alpha³)$. The higher-order solution depends on the perturbation parameter ($\varepsilon$) and the critical environment conditions. The critical environment can be described with higher tide level ($D$), lower hydraulic conductivity ($K$) and wave frequency ($\omega$), which makes higher-order solution significant to be considered.

### 3.2.2 First-order approximation

The governing equation to be solved in the first-order boundary value problem is

$$
2H_{1T} = (H_{0}H_{1})_{XX},
$$

To solve (3.25), $H_{1}$ is expanded as

$$
H_{1} = \sum_{n=1}^{\infty} \alpha^n H_{1n}.
$$

With (3.25) and (3.26), the governing equation and boundary conditions can be sorted in different order of $\alpha$. 

15
\[ O(\varepsilon \alpha) : \quad 2H_{11T} = H_{11XX}, \quad (3.27a) \]
\[ O(\varepsilon \alpha^2) : \quad 2H_{12T} = H_{12XX} + (H_{01}H_{11})_{XX}, \quad (3.27b) \]
\[ H_{11}(0, t) = H_{12}(0, t) = 0 \quad \text{and} \quad H_{11XX}(\infty, t) = H_{12XX}(\infty, t) = 0. \quad (3.27c) \]

The solution of the boundary value problem \( O(\varepsilon \alpha) \) is simply \( H_{11} = 0 \). Then, \((3.27b)\) becomes
\[ 2H_{12T} = H_{12XX}. \quad (3.28) \]
It is obvious that the solution of the boundary value problem \( O(\varepsilon \alpha) \) is \( H_{12} = 0 \), and results \( H_1 = 0 \).

### 3.2.3 Second-order approximation

As mentioned, all previous investigations are limited to the zero-order approximation. The second-order approximation yet has not been considered.

With similar procedure in solving the zero-order, the second-order of water table fluctuation \( H_2 \) can be written as
\[ H_2 = \sum_{n=1}^{\infty} \alpha^n H_{1n}. \quad (3.29) \]

Then, the second-order boundary value problem can be expanded and sorted as
\[ O(\varepsilon \alpha^2) : \quad 2H_{21T} = H_{21XX} + \frac{1}{3}H_{01XXX}, \quad (3.30a) \]
\[ O(\varepsilon^2 \alpha^2) : \quad 2H_{22T} = H_{22XX} + (H_{01}H_{21})_{XX} \]
\[ + \left( H_{01}H_{01XX} + \frac{1}{3}H_{02XX} \right)_{XX}, \quad (3.30b) \]
\[ H_{21}(0, t) = H_{22}(0, t) = 0 \quad \text{and} \quad H_{21XX}(\infty, t) = H_{22XX}(\infty, t) = 0. \quad (3.30c) \]

Following the same procedure as the previous section, the solution is found as
\[ H_{21} = -\frac{X}{3} \exp(-X) [\cos(T - X) + \sin(T - X)], \quad (3.31) \]
\[ H_{22} = -\frac{1}{3} (1 - \exp(-2X)) + \frac{X}{6} \exp(-2X) \]
\[ -\frac{\sqrt{2}X}{3} \exp(-\sqrt{2}X) \cos(2T - \sqrt{2}X) \]
\[ + \frac{1}{3} (1 - \sqrt{2}X) \exp(-\sqrt{2}X) \sin(2T - \sqrt{2}X) \]
\[ + \exp(-2X)[\frac{X}{3} \cos(2T - X) - \frac{1}{3} - \frac{X}{3} \sin(2T - X)]. \quad (3.32) \]
3.3 Results and Discussions

The solution derived for the tide-induced water table fluctuation in a vertical beach is summarized in the following:

\[ H = H_o + \varepsilon H_1 + \varepsilon^2 H_2 + O(\varepsilon^3) \]
\[ = (1 + \alpha H_{01} + \alpha^2 H_{02} + \alpha^3 H_{03}) + \varepsilon^2 (\alpha H_{21} + \alpha^2 H_{22}), \] (3.33)

where \( H_{ij} \) are given in Section 3.3.

Based on this analytical solution, here we investigate the effects of the higher-order components and the important physical parameters including hydraulic conductivity and the relative thickness of the aquifer.

3.3.1 Higher-order components

The new solution presented above is now compared with the existing models. The behavior of the water table fluctuations \( H \) versus time \( (T/2\pi) \) at \( X = 2 \) for various order solutions is illustrated in Figure 3.2. In the figures, the wave frequency \( (\omega) \) and soil porosity \( (n_e) \) are fixed at \( 4\pi/\text{day} \) and 0.35, respectively. In the example, four sets of perturbation parameters \( (\varepsilon, \alpha) \) are used to demonstrate the influence of high-order components. The first parameter, \( \varepsilon \), is the shallow water parameter. Whether or not an oscillation satisfies the shallow water approximation depends on the values of the \( D, n_e, \omega \) and \( K \) in the combination \( Dn_\varepsilon\omega < < 2K \). The second parameter, \( \alpha = A/D \), represents effect of non-linearity in the governing equation. The results are compared through linear solution \( O(\alpha) \), Parlange et al. \( O(\alpha^2) \), Teo et al. \( O(\alpha^3) \) and the present model \( O(\varepsilon^2\alpha^2) \). As shown in the figures, the amplitude parameter \( \alpha \) only affects the tidal fluctuation \( H \) significantly between the first- and second-order solution, i.e., \( O(\alpha) \& O(\alpha^2) \) and \( O(\varepsilon^2\alpha) \& O(\varepsilon^2\alpha^2) \). The difference between Parlange et al. (1984) and Teo et al. (2002) is insignificant.

Figure 3.2 also demonstrate the influence of shallow water parameter \( (\varepsilon) \) on the tide-induced water table fluctuations \( (H) \). In general, the relative difference between zeroth- and second-order solution is significant and increases as shallow water parameter \( (\varepsilon) \) increases. However, the water table fluctuations will be reduced from the higher-order \( \varepsilon \) solution. This implies that the effects of shallow water parameter (i.e., \( \varepsilon \)) will reduce the tidal fluctuation in a coastal aquifer.

To examine further the effects of high-order components, the tidal fluctuation \( (H) \) versus the distance inland \( (X) \) at \( T = 0 \) is illustrated in Figure 3.3. As shown there, the maximum amplitude of the water table level \( (H) \) decreases with inland distance \( (X) \). The influence of the higher-order component \( (\alpha^2 \text{ and above}) \) can be observed in the figure, but no significant difference between \( O(\alpha^2) \) and \( O(\varepsilon\alpha^2) \). The maximum difference between various solution occurs near \( X = 2 \).
Figure 3.2: Comparisons of various solutions for tide-induced water table fluctuations ($H$) versus time ($T$) in a coastal aquifer ($X = 2$).
Figure 3.3: Comparisons of various solutions for the tide-induced water table fluctuations ($H$) versus horizontal distance ($X$) in a coastal aquifer ($T = 0$).
3.3.2 Hydraulic conductivity

Hydraulic conductivity is a measure how fast the pore fluid transfer between porous medium, which is an important parameter in groundwater hydraulics. Thus, it is necessary to investigate its effects on the tidal fluctuations in a coastal aquifer.

Figure 3.4(a) illustrates the effects of hydraulic conductivity ($K$) on the tide-induced water table fluctuations ($H$). In the example, three different values of $K$, 20, 35 and 50 m/day are used. Since hydraulic conductivity only appears in the first-order solution (in non-dimensional form), we only include the solution of $O(\varepsilon\alpha^2)$ in the numerical examples. As shown in the figure, the influence of hydraulic conductivity is more significant near the crest ($T/2\pi$) near 0.25.

To further examine influence of hydraulic conductivity on the water table fluctuation, the the tidal fluctuations ($H$) versus horizontal distance with various $K$ at $T = 0$ is plotted in Figure 3.4(b). As shown in the figure, the influence of $K$ is observed near $X = 2$.

3.3.3 Thickness of aquifer ($D$)

The thickness of aquifer ($D$) is another important parameter involved in the solution. Figure 3.5 illustrates the effects of thickness of aquifer ($D$) on the tidal fluctuations ($H$). As shown in the figures, water table level decreases as $D$ increases.

3.4 Summary

In this chapter, a mathematical solution for tide-induced water table fluctuations adjacent to a vertical beach has been derived in closed-form.

The previous zeroth-order solutions have been extended to higher orders in $\varepsilon$ and $\alpha$. These newly derived solutions demonstrate the significant effects of higher-order components for larger value of the amplitude parameter ($\alpha$) and shallow water parameter ($\varepsilon$). Hydraulic conductivity ($K$) and thickness of aquifer ($D$) also affect the water table fluctuations.

In the following chapter, another mathematical model for a sloping beach will be derived.
Figure 3.4: Tide-induced water table fluctuation \((H)\) versus (a) time \((T)\) and (b) horizontal distance \((X)\) with various hydraulic conductivities \((\alpha = 0.2, D=5\ m)\)
Figure 3.5: Tide-induced water table fluctuation ($H$) versus (a) time ($T$) and (b) $X$ with various aquifer thicknesses ($\alpha = 0.2$, $K = 50$ m/day)
Chapter 4

Tidal Dynamics in a Sloping Beach

The analytical solution presented in Chapter 3 provides some fundamental understanding in tidal dynamics of coastal aquifers adjacent to a vertical beach. In this chapter, a more realistic case of a sloping beach will be considered with the basic assumptions of the fluid similar to the vertical beach model.

Nielsen (1990) was the first to derive an analytical solution for the tide-induced water table fluctuations in a sloping beach. However, there appear to be some difficulties with his solution associated with the definition of his small parameter, and with the satisfactory application of the boundary condition. Furthermore, Nielsen’s (1990) solution only contained part of the higher-order components. These issues will be discussed in this chapter.

Here, a new analytical approach to solve the boundary value problem with a sloping beach is proposed. The concept of a moving boundary will be addressed by introducing a new moving coordinate, which will provide a more realistic boundary condition.

4.1 Boundary Value Problem

The phenomenon of ocean tides incident at a sloping beach is depicted in Figure 4.1. The tides will cause fluctuations of the water table in a coastal aquifer. The horizontal $x$-axis extends positive inland from a fixed origin at the mean sea level. The intersection of the sloping beach and the variable sea level is defined by

$$x_0(t) = A \cot \beta \cos(\omega t), \quad (4.1)$$

where $\beta$ is the slope of a beach. It is noted that the beach is vertical when $\beta=90^\circ$.

At the interface of the ocean and land ($x_0$), the water table level is equal to the ocean wave oscillation, i.e.,

$$h(x_0(t), t) = D + A \cos(\omega t), \quad (4.2)$$

Using the same procedure to non-dimensional the variables as in Chapter 3, the beach face boundary condition (4.2) becomes:

$$H(X_0(T), T) = 1 + \alpha \cos(T), \quad (4.3)$$

where,
Figure 4.1: Schematic drawing of coastal aquifers for sloping beach.

\[ X_0(T) = \alpha \varepsilon \cot \beta \cos(T), \]  
where \( \alpha = A/D \) and \( \varepsilon = D/L \) as defined in Chapter 3.

Since \( X_0(T) \) is a moving boundary, a transformation is introduced so that in the new coordinates the boundary is fixed (Li et al., 2001):

\[ X_1 = X - X_0(T), \] and \( T_1 = T. \]  

Then,

\[ \frac{\partial H}{\partial T} = \frac{\partial H}{\partial T_1} \frac{\partial T_1}{\partial T} + \frac{\partial H}{\partial X_1} \frac{\partial X_1}{\partial T}, \]  
\[ \frac{\partial H}{\partial X} = \frac{\partial H}{\partial X_1}. \]  

Similar to the procedure used in the vertical beach model, the boundary value problem for different orders can be written as

\[ O(1) : 2H_0T_1 = (H_0H_0X_1)_X, \]  
\[ O(\varepsilon) : 2[H_1T_1 + \alpha \sin(T_1) \cot(\beta)H_0X_1] = 2H_0X_1H_1X_1, \]  
\[ + (H_1H_0X_1X_1 + H_0H_1X_1X_1), \]  
\[ O(\varepsilon^2) : 2[H_2T_1 + \alpha \sin(T_1) \cot(\beta)H_1X_1] = \frac{1}{2}(H_1^2)_X X_1, \]  
\[ + (H_0H_2)_X X_1 + \frac{1}{3}(H_0^3H_0X_1X_1)_X X_1, \]

with the boundary conditions

\[ H_0(0, T_1) = 1 + \alpha \cos(T_1), \quad H_1(0, T_1) = H_2(0, T_1) = \cdots = 0, \]  
\[ H_0X_1(\infty, T_1) = H_1X_1(\infty, T_1) = H_2X_1(\infty, T_1) = \cdots = 0. \]

Note that the above governing equations are similar to those for a vertical beach.
4.2 General Solutions

4.2.1 Zeroth-order approximation

From equation (4.8a), the governing equation to be solved for the zeroth-order approximation is given as

\[ 2H_{0T_1} = (H_0 H_{0X_1})_{X_1}. \] (4.11)

Equation (4.11) is identical to the governing equation in a vertical beach. Thus, the solution can be written as

\[ H_0 = 1 + \alpha H_{01} + \alpha^2 H_{02} + O(\alpha^3), \] (4.12a)

\[ H_{01} = \exp(-X_1) \cos(T_1 - X_1), \] (4.12b)

\[ H_{02} = \frac{1}{4} [1 - \exp(-2X_1)] + \frac{1}{2} \{ \exp\left(-\sqrt{2}X_1 \right) \cos(2T_1 - \sqrt{2}X_1) \]
\[ - \exp(-2X_1) \cos(2(T_1 - X_1)). \] (4.12c)

4.2.2 First-order approximation

The governing equation to be solved in the first order boundary value problem is

\[ 2[H_{1T_1} + \alpha \sin(T_1) \cot(\beta) H_{0X_1}] = (H_1 H_0)_{X_1 X_1}, \] (4.13)

To solve (4.13), \( H_1 \) is expanded as

\[ H_1 = \sum_{n=1}^{\infty} \alpha^n H_{1n}, \] (4.14)

With (4.13) and (4.14), the governing equation and boundary conditions can be sorted in different orders of \( \alpha \),

\[ O(\varepsilon \alpha) : 2H_{11T_1} = H_{11X_1 X_1}, \] (4.15a)

\[ O(\varepsilon \alpha^2) : 2H_{12T_1} + 2\sin(T_1) \cot(\beta) H_{01X_1} = H_{12X_1 X_1} + (H_{01} H_{11})_{X_1 X_1}, \] (4.15b)

with boundary conditions

\[ H_{11}(0, T_1) = H_{12}(0, T_1) = 0, \] (4.16a)

\[ H_{11X_1}(\infty, T_1) = H_{12X_1}(\infty, T_1) = 0. \] (4.16b)

The solution of the boundary value problem \( O(\varepsilon \alpha) \) is simply \( H_{11} = 0 \). Then, (4.15b) can be re-written as

\[ 2H_{12T_1} - H_{12X_1 X_1} = -2 \sin(T_1) \cot(\beta) H_{01X_1}. \] (4.17)

The right-hand-side of the above equation can be written as

\[ 2\sin(T_1) \cot(\beta) H_{01Y} = 2 \sin(T_1) \cot(\beta) \{ \sin(T_1 - X_1) - \cos(T_1 - X_1) \} e^{-X_1} \]
\[ = \cot(\beta) [- \sin(2T_1 - X_1) - \sin(X_1) \]
\[ - \cos(2T_1 - X_1) + \cos(X_1)] e^{-X_1} \]
\[ = \sqrt{2} \cot(\beta) e^{-X_1} [\cos(X_1 + \frac{\pi}{4}) - \cos(2T_1 - X_1 - \frac{\pi}{4})]. \] (4.18)
Thus, the solution of (4.17) with (4.16a) and (4.16b) is

\[ H_{12} = \frac{1}{\sqrt{2}} \cot(\beta) \left\{ \left[ \frac{1}{\sqrt{2}} - \exp(-X_1) \cos \left( X_1 - \frac{\pi}{4} \right) \right] 
+ \exp(-\sqrt{2}X_1) \cos(2T_1 - \sqrt{2}X_1 + \frac{1}{4}\pi) 
- \exp(-X_1) \cos(2T_1 - X_1 + \frac{1}{4}\pi) \right\}. \] (4.19)

Note that the order \((\epsilon)\) will vanish when \(\beta = 90\) is inserted into (4.19). That is, this term will not exist for a vertical beach, which is consistent with the solution presented in Chapter 3.

### 4.2.3 Second-order approximation

The governing equation for the second-order boundary value problem is given by

\[ 2 \left[ H_{2X_1} + \alpha \sin(T_1) \cot(\beta) H_{1X_1} \right] = \frac{1}{2} (H_t^2)_{X_1X_1} + (H_0 H_2)_{X_1X_1} + \frac{1}{3} (H_0^3 H_{0X_1X_1})_{X_1X_1} \] (4.20)

Substituting the following expansion into (4.20)

\[ H_0 = 1 + \sum_{n=1}^{\infty} \alpha^n H_{0n}, \quad H_1 = \sum_{n=1}^{\infty} \alpha^n H_{1n}, \quad \text{and} \quad H_2 = \sum_{n=1}^{\infty} \alpha^n H_{2n}, \] (4.21)

the boundary value problems are sorted in terms of \(\alpha\),

\[
O(\alpha) : \quad 2H_{21T_1} = H_{21X_1X_1} + \frac{1}{3} H_{01X_1X_1X_1X_1}, \quad \text{(4.22a)}
\]

\[
O(\alpha^2) : \quad 2H_{22T_1} = H_{22X_1X_1} + (H_{01} H_{21})_{X_1X_1} + \left( H_{01} H_{01X_1X_1} + \frac{1}{3} H_{02X_1X_1} \right)_{X_1X_1}. \quad \text{(4.22b)}
\]

\[
H_{21}(0,T_1) = H_{22}(0,T_1) = 0, \quad \text{(4.23a)}
\]

\[
H_{21X_1}(\infty,T_1) = H_{22X_1}(\infty,T_1) = 0. \quad \text{(4.23b)}
\]

The solutions of the above boundary value problems are

\[
H_{11} = -\frac{\sqrt{2}}{3} X_1 \exp(-X_1) \cos(T_1 - X_1 - \frac{\pi}{4}), \quad \text{(4.24a)}
\]

\[
H_{22} = -\frac{1}{3} + \frac{1}{6} (2 + X_1) \exp(-2X_1) \left[ -\frac{2}{3} X_1 e^{-\sqrt{2}X_1} \cos(2T_1 - \sqrt{2}X_1 - \frac{\pi}{4}) + \frac{1}{3} e^{-\sqrt{2}X_1} \sin(2T_1 - \sqrt{2}X_1) 
+ \frac{\sqrt{2}}{3} X_1 e^{-2X_1} \cos(2T_1 - 2X_1 - \frac{\pi}{4}) - \frac{1}{3} e^{-2X_1} \sin(2T_1 - 2X_1) \right]. \quad \text{(4.24b)}
\]
4.2.4 Special case: a vertical beach

In summary, the solution of tide-induced water table fluctuations in a coastal aquifer can be written as

\[ H(X_1, T_1) = \alpha e^{-X_1} \cos(\theta_1) \]

\[ + \alpha^2 \left\{ \frac{1}{4} (1 - e^{-2X_1}) + \frac{1}{2} [e^{-\sqrt{2}X_1} \cos(\theta_2) - e^{-2X_1} \cos(2\theta_1)] \right\} \]

\[ + \frac{1}{\sqrt{2}} \cot(\beta) \varepsilon \alpha^2 \left\{ \frac{1}{\sqrt{2}} - e^{-X_1} \cos \left( X_1 - \frac{1}{4}\pi \right) \right\} \]

\[ + e^{-\sqrt{2}X_1} \cos(\theta_2 + \frac{1}{4}\pi) - e^{-X_1} \cos(\theta_3 + \frac{1}{4}\pi) \]

\[ - \frac{\sqrt{2}}{3} \varepsilon^2 \alpha X e^{-X_1} \cos(\theta_1 - \frac{\pi}{4}) \]

\[ + \frac{1}{3} \varepsilon^2 \alpha^2 \left\{ -1 + \left( 1 + \frac{X}{2} \right) e^{-2X_1} - 2Xe^{-\sqrt{2}X_1} \cos(\theta_2 - \frac{\pi}{4}) \right\} \]

\[ + e^{-\sqrt{2}X_1} \sin(\theta_2) + \sqrt{2}X e^{-2X_1} \cos(2\theta_1 - \frac{\pi}{4}) - e^{-2X_1} \sin(2\theta_1) \}, \]

(4.25)

where \( \theta_1 = T_1 - X_1, \quad \theta_2 = 2T_1 - \sqrt{2}X_1, \quad \) and \( \theta_3 = 2T_1 - X_1. \)

For the special case of a vertical beach, \( \beta = 90^\circ, \) leading to \( X_1 = X, \) \( T_1 = T \) and \( \cot\beta = 0, \) (4.25) becomes

\[ H(X, T) = \alpha e^{-X} \cos(\theta_1) \]

\[ + \alpha^2 \left\{ \frac{1}{4} (1 - e^{-2X}) + \frac{1}{2} [e^{-\sqrt{2}X} \cos(\theta_2) - e^{-2X} \cos(2\theta_1)] \right\} \]

\[ - \frac{\sqrt{2}}{3} \varepsilon^2 \alpha X e^{-X} \cos(\theta_1 - \frac{\pi}{4}) \]

\[ + \frac{1}{3} \varepsilon^2 \alpha^2 \left\{ -1 + \left( 1 + \frac{X}{2} \right) e^{-2X} - 2Xe^{-\sqrt{2}X} \cos(\theta_2 - \frac{\pi}{4}) \right\} \]

\[ + e^{-\sqrt{2}X} \sin(\theta_2) + \sqrt{2}X e^{-2X} \cos(2\theta_1 - \frac{\pi}{4}) - e^{-2X} \sin(2\theta_1) \}, \]

(4.26)

which is identical to the solution of \( O(\varepsilon^2\alpha^2) \) for a vertical beach given in Chapter 3.

### 4.3 Results and Discussions

A new analytical solution for tide-induced water table fluctuations in a sloping beach is presented in the previous sections. As mentioned previously, Nielsen’s (1990) result is the only analytical solution available for the case of a sloping beach. Here, we will make a comprehensive comparison between Nielsen (1990) and the present solution. Also, since the fundamental characteristics of the tidal dynamics in coastal aquifers have been discussed in Chapter 3, we only examine the effect of beach slope \( (\beta) \) on the water table fluctuations.
4.3.1 Comparison with Nielsen (1990)

The solution presented by Nielsen (1990) is the only previous analytical solution available for tidal dynamics in a sloping beach. The solution is summarised in a non-dimensional form as below:

\[ H_{\text{Nielsen}} = 1 + \alpha e^{-X} \cos(T - X) + \alpha \varepsilon_N \left[ \frac{1}{2} + \frac{\sqrt{2}}{2} \cos(2T - \sqrt{2}X + \frac{\pi}{4}) \right], \]  

(4.27)

where

\[ \varepsilon_N = \frac{A}{L} \cot(\beta) = \alpha \varepsilon \cot(\beta) \]  

(4.28)

From the above solution, Nielsen (1990) used \( \varepsilon_N \) as the perturbation parameter. Clearly, when the slope \( \beta = 90^\circ \), \( \varepsilon_N = 0 \), which leads to the linear solution for a vertical beach.

Before we compare Nielsen (1990) and the present solution numerically, we investigate (4.27) and (4.28) first. From (4.27) and (4.28), we observed few shortcomings in Nielsen (1990) solution:

1. **The perturbation parameter (\( \varepsilon_N \)):** The perturbation used in Nielsen includes the slope of the beach (\( \beta \)) limiting the applicable range of the solution. Since \( \varepsilon_N \) is a perturbation parameter, assumed to be less than unity, the slope of the beach (\( \beta \)) should satisfy \( \tan^{-1}(\alpha \varepsilon) < \beta < \pi/2 \). For example, \( 26.6^o < \beta < 90^o \) when \( \alpha \varepsilon = 0.5 \), and \( 11.3^o < \beta < 90^o \) for \( \alpha \varepsilon = 0.2 \).

2. **Incomplete solution for higher-order components:** Comparing (4.25) and (4.27), it is observed that Nielsen’s solution is only part of the present solution up to \( O(\varepsilon \alpha^2) \). His solution does not include the oscillating term, \( e^{-X_1} \cos(\theta_3 + \frac{1}{4} \pi) \) and non-oscillating term, \( e^{-X_1} \cos(X_1 - \frac{1}{4} \pi) \). Thus, Nielsen’s solution is an incomplete solution of \( O(\varepsilon \alpha^2) \).

3. **Boundary condition at \( X_1 = 0 \) (i.e., \( X = X_0 \)):** It is clear that (4.25) satisfies the boundary condition at the intersection between ocean and coastal aquifer, i.e., (4.3). However, Nielsen’s solution (4.27) does not satisfy (4.3). In fact, Nielsen’s solution will only satisfy the boundary condition when all terms of the perturbation expansion are included.

To further investigate the difference between Nielsen (1990) and the present solutions, the distribution of water table fluctuations for various beach slopes are illustrated in Figure 4.2. The slope of the beach (\( \beta \)) varies between 15 and 60 degrees. As shown in the figure, Nielsen’s (1990) solution appears to be in between \( O(\varepsilon \alpha^2) \) and \( O(\alpha) \) of the present solutions. This can be due to the missing term in Nielsen’s solution, as discussed previously. It is also observed that for larger values of \( \varepsilon \) the water table fluctuations will decrease as a result of negative sign in the term of \( O(\varepsilon^2) \) in (4.25).

Figure 4.3 illustrates the water table fluctuation in the horizontal direction at \( T = 0 \). It is obvious that Nielsen (1990) solution does not satisfy the boundary condition \( H(X_0, T_1) = 1 + \alpha \cos(T_1) \), while the present solution satisfy the boundary condition. In the example, \( H(0, 0) = 1.35 \) with \( \alpha = 0.35 \). This numerical comparison further confirm the previous discussion.
Figure 4.2: Comparison of tide-induced water table fluctuations in a sloping beach ($\varepsilon=0.5 \alpha = 0.35, X=1$)
Figure 4.3: Comparison of tide-induced water table fluctuations in a sloping beach ($\varepsilon=0.5$ $\alpha=0.35$, $T=0$)
4.3.2 Effects of beach slopes ($\beta$)

The fundamental characteristics of tide-induced water table fluctuations in a coastal aquifer have been discussed in Chapter 3 for a vertical beach. Here, we focus on the effects of beach slope ($\beta$). In general, the slope variation at the beach would vary depends on the dynamics mechanism effects from the ocean, and the properties of the soil itself; such as the hydraulic conductivity, soil porosity, cohesiveness and the soil particle distribution. The present solution can be used to predict the groundwater fluctuations in a sandy beach for different input of slopes.

Figures 4.4 illustrates the tidal fluctuations in both time and space. As shown in the figure, the water table level increases as the slope increases. This is because the component of $O(\varepsilon \alpha^2)$ will become more significant due to increasing $\cot(\beta)$.

It can be observed in Figure 4.4 that different beach slope results to different fluctuations of watertable in a period of time. At different degrees of sloping, the time scale in between crest and trough will vary which result to phase lag. As the slope reduces, a wider time scale in between crest-trough can be seen and this will also result to higher fluctuations of watertable.

4.4 Summary

In this chapter, a new analytical solution for tide-induced water table fluctuations adjacent to a sloping beach has been derived. The solution has been solved up to higher orders in $\varepsilon$ and $\alpha$. The present solution reduces directly to that for a vertical beach when $\beta = 90^\circ$.

Comprehensive comparisons between Nielsen’s (1990) solution and the present solution have been performed mathematically and numerically. The shortcoming of Nielsen (1990) solution have been clearly indicated, and the present solution has been demonstrated to provide a better prediction.

The new solution also demonstrates the significant influence of beach slope ($\beta$) on the water table fluctuations. In general, the water table level will increase as the slope of a beach decreases.
Figure 4.4: Tide-induced water table fluctuations in a sloping beach with various slopes of beaches ($\varepsilon=0.5 \alpha=0.35$)
Chapter 5

Conclusions

5.1 Conclusions

The construction of analytical solutions for tide-induced water table fluctuations in coastal groundwater is challenging. This is due to the involvement of tidal-groundwater interactions, and the phenomena of hydrodynamic dispersion between salt-fresh water in the coastal region. Since the 1950’s, the Boussinesq equation has been fundamental for an understanding of such problems. However, the use of the Boussinesq equation is over simplified in many situations.

In this study two new solutions have been developed for an unconfined coastal aquifer. The solutions are presented in Chapters 3 and 4. Solutions for both vertical and sloping beaches have been derived based on a perturbation technique up to higher orders in $\varepsilon$ and $\alpha$. Based on examples considered, the following conclusions can be drawn.

1. The higher-order analytical solution for tidal dynamics to a vertical beach derived in Chapter 3 is an extension of the zeroth-order solution proposed by Parlange et al. (1994). The present second-order solution demonstrates the significant influence of the higher-order components on the water table level in the case of higher frequency, thicker aquifers or low hydraulic conductivity.

2. Numerical examples indicate that the relevant difference between Parlange et al. (1984) $[O(\alpha^2)]$ and Teo et al. (2002) $[O(\alpha^3)]$ is insignificant, while significant differences between the previous work and the present solution are observed.

3. A new approach is proposed to overcome the shortcomings of the previous solution for a sloping beach. It has been demonstrated in Chapter 4 that the present solution provides a complete higher-order solution, which significantly affects the water table fluctuations. The results indicate that the water table level will increase as the slope of the beach decreases.
5.2 Recommendations for Future Research

The strength of the present study is in developing a series of analytical solutions for the tide-induced water table fluctuations in a coastal aquifer. Although the present study has covered many applications for the tidal dynamics, the following topics require further research.

1. The capillary fringe is not considered in the present study. That is, here the upper boundary of the flow domain is a free surface which is a sharp boundary between saturated and dry aquifer material. In more realistic conditions the upper boundary is not a sharp air-water interface, but rather a gradual and diffuse transition zone of partly saturated material. Therefore, it is important to correct the ideal assumption, and take into account the capillary fringe. The present framework can be further extended for such a condition.

2. It is noted that most previous investigations have been limited to one-dimensional models, due to the complicated mechanism. Only a few researchers attempted to solve the problem through two-dimensional approaches, and only a vertical beach has been considered. Thus, a more advanced two-dimensional analysis for a sloping beach is desired.

3. The hydraulic conductivity is assumed to be a constant in the present study. However, it may vary with depth or inland distance. Thus, a possible solution for the tidal dynamics in non-homogeneous coastal aquifers is desired.
References


Appendix: Mathematical Derivation of Boundary Value Problem

In this appendix, the mathematical details of the governing equation are given here. The boundary conditions apply to the Laplace governing equation leads to

\[ \Phi_0 = C_0(X, T), \quad (A.1a) \]
\[ \Phi_1 = C_1(X, T), \quad (A.1b) \]
\[ \Phi_2 = C_2(X, T) - \frac{Z^2}{2} \Phi_{0XX}, \quad (A.1c) \]
\[ \Phi_3 = C_3(X, T) - \frac{Z^2}{2} \Phi_{1XX}, \quad (A.1d) \]
\[ \Phi_4 = C_4(X, T) - \frac{Z^2}{2} C_{2XX} + \frac{Z^4}{24} \Phi_{0XXX}. \quad (A.1e) \]

where

\[ C_2(X, T) = H_2 + \frac{G^2}{2} \Phi_{oXX}, \quad (A.2a) \]
\[ C_{2X} = H_{2X} + \frac{1}{2} (H^2 \Phi_{oXX})_X \]
\[ = H_{2X} + \frac{1}{2} (2HHX \Phi_{oXX} + H^2 \Phi_{oXXX}), \quad (A.2b) \]
\[ C_{2XX} = H_{2XX} + \frac{1}{2} (H^2 \Phi_{oXX})_{XX} \]
\[ = H_{2XX} + H_X^2 \Phi_{oXX} + HH_{XX} \Phi_{oXX} + 2HHX \Phi_{oXXX} + \frac{1}{2} H^2 \Phi_{oXXXXX}. \quad (A.2c) \]

With boundary condition, as the \( \Phi = H \), we can define different order watertable height in the relation as follow,

\[ \Phi_o = H_o, \quad (A.3a) \]
\[ \Phi_1 = H_1, \quad (A.3b) \]
\[ \Phi_2 = H_2, \quad (A.3c) \]
\[ \Phi_3 = H_3, \quad (A.3d) \]
\[ \Phi_4 = H_4, \quad (A.3e) \]
With the above relations, the equations can be rewritten as:

\[
(\Phi_X|_H)^2 = \left[ \Phi_oX + \varepsilon \Phi_1X + \varepsilon^2 \left( C_{2X} - \frac{1}{2}(H^2\Phi_{0XX})X \right) \right]^2
= H_0^2 + +2\varepsilon H_oX H_1X + \varepsilon^2[H_{1X}^2 + 2H_oX (H_{2X} + H_oH_oX H_{0XX})],
\]

\[
\Phi_Z|_H = -\varepsilon^2 Z\Phi_{oXX} - \varepsilon^3 Z\Phi_{1XX} - \varepsilon^4 \left( ZC_{2XX} - \frac{1}{6}H^3\Phi_{0XXX} \right) \mid H
= -\varepsilon^2 H_oX H_{0XX} - \varepsilon^3 (H_1H_oXX + H_0H_{1XX})
- \varepsilon^4 \{H_2H_oXX + H_1H_1XX + H_oH_{2XX} + H_o^2H_{0XX}
+ H_0^2H_oXX H_o + 2H_o^2H_oX H_{0XXX} + \frac{1}{3}H_o^3H_{0XXX} \},
\]

\[
(\Phi_Z|_H)^2 = \varepsilon^4 H_o^2 H_{0XX}^2.
\]

\[
\Phi_T|_H = H_oT + \varepsilon (H_1T + \alpha \sin(T) \cot(\beta) H_oX)
+ \varepsilon^2 [H_2T + H_oH_oT H_{0XX} + \alpha \sin(T) \cot(\beta) H_{1X}].
\]

Hence, yields the governing equation in different order for sloping beach in orders of \( \varepsilon \).

\[
O(1) : \quad 2H_oT = (H_oH_oX)_X, \tag{A.4a}
\]

\[
O(\varepsilon) : \quad 2[H_1T + \alpha \sin(T) \cot(\beta) H_{0X}] = (H_oH_1)_{XX}, \tag{A.4b}
\]

\[
O(\varepsilon^2) : \quad 2[H_2T + \alpha \sin(T) \cot(\beta) H_{1X}] = \frac{1}{2}(H_1^2)_{XX}
+ (H_0H_2)_{XX} + \frac{1}{3}(H_o^3H_{0XX})_{XX}. \tag{A.4c}
\]

For the special case, a vertical beach, \( \cot(\beta) = 0 \). Then, the non-linear phreatic free surface kinematic boundary conditions for vertical beach can be reduced to

\[
O(1) : \quad 2H_oT = (H_oH_oX)_X, \tag{A.5a}
\]

\[
O(\varepsilon) : \quad 2H_1T = (H_oH_1)_{XX}, \tag{A.5b}
\]

\[
O(\varepsilon^2) : \quad 2H_2T = (H_oH_2)_{XX} + \frac{1}{3}(H_o^3H_{0XX})_{XX}. \tag{A.5c}
\]